### (D) Formula for Pedal Equations.

*To find the radius of curvature for the pedal curve*

$$p = f(r).$$

(MU 1971, '77, '79)

or, *To prove the formula* $\rho = r\dfrac{dr}{dp}$, *where the symbols have their usual meaning.*

(MU 1990; PU '66, '63, '61, '60; Mlth U '82; RU '65, '64; Bh U '86; BU '67)

Let $P$ be any point $(r, \theta)$ on the curve.

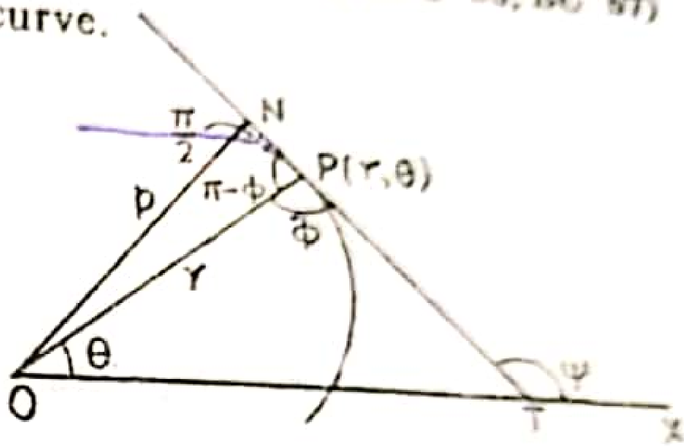
Then $OP = r$, $\angle POX = \theta$.

Draw the tangent $PT$ to the curve at $P$.

Draw $ON$ perpendicular to this tangent.

Let $ON = p$, $\angle OPT = \varphi$,

$\angle OPN = (\pi - \varphi)$, $\angle PTX = \psi$




We have $\sin\varphi = r\dfrac{d\theta}{ds}$ and $\cos\varphi = \dfrac{dr}{ds}$. ... (1)

Clearly

$$\psi = \theta + \varphi.$$

Differentiating with respect to $s$, we get $\dfrac{d\psi}{ds} = \dfrac{d\theta}{ds} + \dfrac{d\varphi}{ds}$

i.e.,

$$\frac{1}{\rho} = \frac{d\theta}{ds} + \frac{d\varphi}{ds}. \quad \dots (2)$$

From the right-angled $\triangle ONP$, $\sin(\pi - \varphi) = \dfrac{ON}{OP}$

i.e.,

$$\sin\varphi = \frac{p}{r}$$

i.e., $$p = r \sin \phi.$$

Differentiating both sides with regard to $r$, we get

$$\frac{dp}{dr} = \sin \phi + r \cos \phi \frac{d\phi}{dr}$$

$$= r \frac{d\theta}{ds} + r \cdot \frac{dr}{ds} \cdot \frac{d\phi}{dr}, \text{ from (1)}$$

$$= r \left( \frac{d\theta}{ds} + \frac{d\phi}{ds} \right) = r \cdot \frac{1}{\rho}, \text{ from (2).}$$

Hence $$\boxed{\rho = r \frac{dr}{dp}.}$$

*Illustration.* Prove that $\rho \propto p$ in case of an epicycloid whose equation is

$$p^2 = Ar^2 + B.$$

*Solution.* The equation of an epicycloid is

$$p^2 = Ar^2 + B. \quad \ldots (1)$$

Differentiating (1) with respect to $r$, we get

$$2p \frac{dp}{dr} = 2Ar; \quad \text{or} \quad r \frac{dr}{dp} = \frac{p}{A}.$$

We know that $$\rho = r \frac{dr}{dp}$$

or $$\rho = \frac{p}{A}.$$

Hence $$\rho \propto p.$$

**(E) Formula for Polar Equation.**

(i) *To find the radius of curvature for the polar curve* $r = f(\theta)$.

**(MU 1971, '74, '78, '81, '69, '67, '65; Bh U '69, '68, '66, '64, '63; RU '85; BPSC '78)**

or, *To prove the formula :* $\rho = \dfrac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2}$, *where the symbols have their usual meaning.*

**(PU 1964, '83; Bh U '84; BU '84)**

**Proof.** We know that $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left( \frac{dr}{d\theta} \right)^2. \quad \ldots (1)$$

Differentiating both sides with regard to $\theta$, we get

$$-\frac{2}{p^3} \cdot \frac{dp}{d\theta} = -\frac{2}{r^3} \cdot \frac{dr}{d\theta} - \frac{4}{r^5} \left( \frac{dr}{d\theta} \right)^3 + \frac{2}{r^4} \cdot \frac{dr}{d\theta} \cdot \frac{d^2 r}{d\theta^2}$$

or $$\frac{dp}{dr}=\left\{\frac{1}{r^3}+\frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^2-\frac{1}{r^4}\cdot\frac{d^2r}{d\theta^2}\right\}p^3$$

$$=\left\{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}\right\}\cdot\frac{p^3}{r^5}. \qquad \dots (2)$$

From (1), $$\frac{1}{p^2}=\frac{r^2+\left(\frac{dr}{d\theta}\right)^2}{r^4}$$

or $$p=\frac{r^2}{\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}}. \qquad \dots (3)$$

We know that $$\rho=r\frac{dr}{dp}$$

$$=r.\frac{r^5}{p^3\left\{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}\right\}}, \text{ from (2)}$$

$$=\frac{r^6.\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^6\left\{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}\right\}}.$$

Hence $$\boxed{\rho=\frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}.}$$

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2. (a) Find the radius of curvature for the curve

$y=4\sin x-\sin 2x$ at the point $x=\frac{\pi}{2}$. (MU 1968)

*Solution.* We have $y=4\sin x-\sin 2x$.

Differentiating with respect to $x$, we get

$$y_1=4\cos x-2\cos 2x.$$

Again differentiating with respect to $x$, we get

$$y_2=-4\sin x+4\sin 2x.$$

At $$x=\frac{\pi}{2},\ y_1=4\cos\frac{\pi}{2}-2\cos\pi=2$$

and $$y_2=-4\sin\frac{\pi}{2}+4\sin\pi=-4.$$

$$\therefore\ \rho=\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}=\frac{(1+4)^{\frac{3}{2}}}{-4}=\frac{5\sqrt{5}}{4},$$ considering its magnitude.

2. (b) If $\rho$ be the radius of curvature of a parabola at a point whose distance measured along the curve from a fixed point is $s$, prove that

$$3\rho\frac{d^2\rho}{ds^2}-\left(\frac{d\rho}{ds}\right)^2-9=0.$$

(MU 1969 H, '70 H; RU '83; Bh U '63 H, '65 H; BU '67 H; PU '44 H, '56 H, '61 H)

**Solution.** Let the equation of the parabola be

$$y^2=4ax;\ \text{or}\ y=2\sqrt{ax}. \qquad \ldots (1)$$

Differentiating successively with respect to $x$, we get

$$y_1=\frac{\sqrt{a}}{\sqrt{x}};\ y_2=-\frac{1}{2}\cdot\frac{\sqrt{a}}{x^{\frac{3}{2}}}.$$

$$\therefore\ \rho=\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}=\frac{2}{\sqrt{a}}(x+a)^{\frac{3}{2}},$$ in magnitude.

That is why the formula $\rho = \frac{(1+x_1^2)^{\frac{3}{2}}}{x_2}$ will be taken into account.

Now $$y^2 = 16\left(1+\frac{4}{x}\right).$$

Differentiating successively with respect to $y$, we get

$$2y = -\frac{64}{x^2}\cdot\frac{dx}{dy}; \quad \text{or} \quad \frac{dx}{dy} = -\frac{x^2 y}{32}. \qquad \text{(1)}$$

Again $$\frac{d^2x}{dy^2} = -\frac{1}{32}\left(2xy\frac{dx}{dy}+x^2\right).$$

At the point $(-4, 0)$,

$$\frac{dx}{dy} = 0, \ \frac{d^2x}{dy^2} = -\frac{1}{2}.$$

Hence, at the point $(-4, 0)$,

$$\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}} = \frac{1}{-\frac{1}{2}} = -2.$$

$\therefore \quad \rho = 2$, in magnitude.